Exact SNR and SIR analysis in Poisson wireless networks

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The probability density function and cumulative distribution function of the received signal-to-noise ratio (SNR) and the received signal-to-interference ratio (SIR), for interference-limited systems is derived, at the *n*th nearest neighbour node in a Poisson point process wireless random network. The analytical expressions are given in terms of the Meijer G-function and reveal the impact of node spatial density, transmit power, interference power, and path-loss exponent on the connectivity probability of a broadcast wireless transmission. The analytical results are validated with computer simulation.

Introduction: Many works studied distance distributions in random infinite and finite networks [1–3]. A model that is usually used to characterise the spatial distribution of these nodes is the Poisson point process (PPP). For example, the work in [4] studied the connectivity probability for vehicular-to-vehicular (V2V) and vehicular-to-infrastructure communication scenarios under the assumption that vehicles are distributed on the road following a Poisson distribution. For a random network with node distribution based on the PPP model and the transmitter located at the origin, the distance between this node and its *n*th nearest neighbour is a random variable that follows the generalised Gamma distribution [1, 5]. Therefore, for constant transmit power and a power-law path-loss model, the received average signal-to-noise ratio (SNR) or the average received signal-to-interference ratio (SIR), for the case of interference-limited systems, are random variables.

In this Letter, we consider the combined effect of path-loss and Nakagami-*m* fading on the outage probability of the *n*th nearest neighbour node. Closed-form expressions for the probability density function (PDF) and cumulative distribution function (CDF) of the received SNR or received SIR at the *n*th nearest neighbour are derived. The analytical results can be utilised to determine the probability of correct detection at the *n*th nearest neighbour node in a PPP wireless network with different node densities.

Signal to noise analysis: The Poisson process is suitable for modelling uniformly random networks. The Euclidean distance between a point at the origin and its *n*th nearest neighbour, R_n, is distributed according to the generalised Gamma distribution [1]:

$$f_{R_n}(r) = \frac{m(\lambda c_m r^m)^n}{r\Gamma(n)} \exp(-\lambda c_m r^m) \tag{1}$$

where λ is the node spatial density, $c_m r^m$ is the volume of the m-dimensional ball of radius r, c_m is given by

$$c_m = \begin{cases} \dfrac{\pi^{m/2}}{(m/2)!}, & \text{even } m \\ \dfrac{\pi^{(m-1)/2} 2^m ((m-1)/2)}{m!} & \text{odd } m \end{cases} \tag{2}$$

and $\Gamma(n)$ is the gamma function evaluated at n. In addition, if we want to consider only neighbouring nodes that lie within a sector with opening angle ϕ, this simply corresponds to a change of the volume from an m-ball to an m-sector (with opening angle ϕ) whose volume is $c_{\phi,m} r^m$. Therefore, the PDF of the distance to the nth nearest neighbour in a sector ϕ is given by replacing c_m by $c_{\phi,m}$ in (2). For $m = 1, 2, 3$, we have $c_{\phi,1} = 1$, $c_{\phi,2} = \phi$, and $c_{\phi,3} = (2\pi/3)(1 - \cos(\phi))$, respectively. Moreover, it is assumed that the transmitted signal undergoes small-scale Nakagami-*m* fading. The corresponding instantaneous received SNR, X, follows the gamma distribution with PDF

$$f_X(x) = \left(\frac{m_s}{\Omega_s}\right)^{m_s} \frac{x^{m_s-1}}{\Gamma(m_s)} \exp\left(-\frac{m_s}{\Omega_s} x\right) \tag{3}$$

where m_s and Ω_s are the distribution's shaping and scaling parameters. When the path-loss follows the decaying power law, the average SNR Ω_s at distance r is given by

$$\Omega_s = P_t K r^{-\alpha}/N = \tilde{P}_t r^{-\alpha} \tag{4}$$

where P_t is the transmit power, K is a constant that depends on the antenna characteristics and free-space path-loss up to distance $r_0 = 1$ m, r is a random variable which follows the distribution in (1), α is the path-loss exponent with values in the range [2, 6], N is the receiver noise power, and $\tilde{P}_t$ is the transmit SNR. The PDF of the received SNR at the nth nearest neighbour is then given by

$$\begin{aligned} f_{X_n}(x) &= \int_0^\infty f_X(x|r) f_{R_n}(r)\,\mathrm{d}r \\ &= \frac{x^{m_s-1} m}{\Gamma(m_s)\Gamma(n)} \left(\frac{m_s}{\tilde{P}_t}\right)^{m_s} (\lambda c_{\phi,m})^n \\ &\quad \times \int_0^\infty r^{\alpha m_s + mn - 1} \exp\left(-\frac{m_s x}{\tilde{P}_t} r^\alpha\right) \exp(-\lambda c_{\phi,m} r^m)\,\mathrm{d}r \end{aligned} \tag{5}$$

The previous integral can be solved using the following result, which is derived from [6, Equations (11) and (21)] as

$$\begin{aligned} \int_0^\infty x^\nu \exp(-B_1 x^{b_1}) \exp(-B_2 x^{b_2})\,\mathrm{d}x &= \frac{1}{\sqrt{b_1 b_2}} \left(\frac{b_2}{B_1}\right)^{((\nu+1)/b_1)} \\ &\quad \times \frac{1}{(2\pi)^{((b_1+b_2-2)/2)}} G^{b_1 b_2}_{b_2 b_1}\left(\frac{B_2^{b_1} b_2^{b_2}}{B_1^{b_2} b_1^{b_1}} \middle| \begin{matrix} \Delta\left(b_2, 1 - \frac{\nu+1}{b_1}\right) \\ \Delta(b_1, 0) \end{matrix} \right) \end{aligned} \tag{6}$$

where b_1 and b_2 are integers, $G(\cdot)$ is the Meijer's G-function and $\Delta(m, n) = n/m, \ldots, ((n + m - 1)/m)$. It is noted that Meijer G-functions are built-in functions in many mathematical software packages, e.g. Mathematica, Maple, and thus can be directly evaluated. For path-loss exponent expressed as $\alpha = \ell/k$, where ℓ and k are integers, based on the solution given in (6) with $b_1 = \ell$ and $b_2 = km$, the PDF of the SNR at the nth nearest neighbour node is given in closed form as

$$\begin{aligned} f_{X_n}(x) &= \frac{(m_s/\tilde{P}_t)^{m_s} (\lambda c_{\phi,m})^n x^{m_s-1}}{\Gamma(m_s)\Gamma(n)(2\pi)^{((\ell+km)/2)-1}} \sqrt{\frac{m}{\alpha}} \left(\frac{m_s x}{km\tilde{P}_t}\right)^{-m_s-(mn/\alpha)} \\ &\quad \times G^{\ell,km}_{km,\ell}\left(\left(\frac{\lambda c_{\phi,m}}{\ell}\right)^\ell \left(\frac{km\tilde{P}_t}{m_s x}\right)^{km} \middle| \begin{matrix} \Delta\left(km, 1 - m_s - \frac{mn}{\alpha}\right) \\ \Delta(\ell, 0) \end{matrix} \right) \end{aligned} \tag{7}$$

Moreover, using [6, eq. (26)], the corresponding CDF of the received SNR is given in closed form as

$$\begin{aligned} F_{X_n}(x) &= \int_0^x f_{X_n}(x)\,\mathrm{d}x \\ &= 1 - \frac{(m_s/\tilde{P}_t)^{m_s} (\lambda c_{\phi,m})^n}{\Gamma(m_s)\Gamma(n) k (2\pi)^{((\ell+km)/2)-1} \sqrt{\alpha m}} \left(\frac{m_s}{km\tilde{P}_t}\right)^{-m_s-(mn/\alpha)} \\ &\quad \times \left(\frac{1}{x}\right)^{mn/\alpha} G^{\ell,km+1}_{km+1,\ell+1}\left(\left(\frac{\lambda c_{\phi,m}}{\ell}\right)^\ell \left(\frac{km\tilde{P}_t}{m_s x}\right)^{km} \right. \\ &\quad \left. \middle| \begin{matrix} \Delta\left(km, 1 - m_s - \frac{mn}{\alpha}\right), 1 - \frac{n}{\ell} \\ \Delta(\ell, 0), -\frac{n}{\ell} \end{matrix} \right) \end{aligned} \tag{8}$$

Signal to interference analysis: Assuming that the received signal is subject to interference, the received SIR for an interference-limited system is given by

$$\gamma_{out} = \frac{X}{Y} \tag{9}$$

where Y denotes the instantaneous received interference-to-noise ratio (INR). Assuming a single dominant Nakagami-*m* interferer, the PDF of γ_{out} is given by

$$f_{\gamma_{out}}(x) = \left(\frac{m_s}{\Omega_s}\right)^{m_s} \left(\frac{m_I}{\Omega_I}\right)^{m_I} \frac{\Gamma(m_s + m_I)}{\Gamma(m_s)\Gamma(m_I)} \frac{x^{m_s-1}}{\left(\frac{m_I}{\Omega_I} + \frac{m_s}{\Omega_s} x\right)^{m_s+m_I}} \tag{10}$$

where Ω_I is the average INR and m_I is the Nakagami fading parameter of the interference signal. Based on (10), using [6, eqs. (10), (11) and (21)] the PDF of the SIR at the nth nearest neighbour is given by

$$f_{\gamma_{\text{out},n}}(x) = \frac{k^{m_s+m_I+(\alpha m_s/m)+n-1/2}}{\Gamma(m_s)\Gamma(m_I)\Gamma(n)}\left(\frac{m_s\Omega_I}{m_I\tilde{P}_t}\right)^{m_s} \times \frac{m^{m_s+m_I}\alpha^{(\alpha m_s/m)+n-1/2}}{(2\pi)^{(\ell/2)+km-3/2}\left(\lambda c_{\phi,m}\right)^{\alpha m_s/m}}x^{m_s-1} \times G^{km,\ell+km}_{\ell+km,km}\left(\left(\frac{\ell}{\lambda c_{\phi,m}}\right)^{\ell}\left(\frac{m_s\Omega_I x}{m_I\tilde{P}_t}\right)^{km} \middle| \begin{matrix}\Delta(km,1-m_s-m_I),\Delta\left(\ell,1-n-\frac{\alpha m_s}{m}\right)\\ \Delta(km,0)\end{matrix}\right) \quad (11)$$

Moreover, using [6, eq. (26)], the corresponding CDF of the received SIR is given in closed form as

$$F_{\gamma_{\text{out},n}}(x) = \frac{k^{m_s+m_I+(\alpha m_s/m)+n-3/2}}{\Gamma(m_s)\Gamma(m_I)\Gamma(n)}\left(\frac{m_s\Omega_I}{m_I\tilde{P}_t}\right)^{m_s} \times \frac{m^{m_s+m_I-1}\alpha^{(\alpha m_s/m)+n-1/2}}{(2\pi)^{\ell/2+km-3/2}\left(\lambda c_{\phi,m}\right)^{\alpha m_s/m}}x^{m_s} \times G^{km,\ell+km+1}_{\ell+km+1,km+1}\left(\left(\frac{\ell}{\lambda c_{\phi,m}}\right)^{\ell}\left(\frac{m_s\Omega_I x}{m_I\tilde{P}_t}\right)^{km} \middle| \begin{matrix}\Delta(km,1-m_s-m_I),\Delta\left(\ell,1-n-\frac{\alpha m_s}{m}\right),1-\frac{m_s}{km}\\ \Delta(km,0),-\frac{m_s}{km}\end{matrix}\right) \quad (12)$$

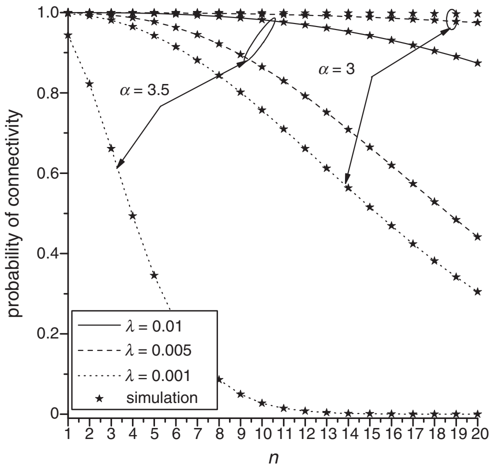

Fig. 1 *Probability of connectivity based on SNR versus neighbour index n*

Numerical results: The probability of coverage or probability of connectivity for the nth nearest neighbour can be defined as the complementary CDF of the received SNR as

$$\Pr\{X_n > \gamma_{th}\} = 1 - F_{X_n}(\gamma_{th}) \quad (13)$$

which is the probability that the SNR at the nth nearest neighbour is greater than the target SNR γ_{th}.

For all the numerical results, we consider a 2D case and a sector of 90° angle, by setting $m = 2$, $\phi = \pi/4$ and $c_{\phi,2} = \phi$. This setting could model a V2V communications scenario where a car broadcasts information to the vehicles behind it. To determine the coverage area of the transmitter, the average distance to the nth nearest neighbour, is given by [1]

$$E[d_n] = \frac{\Gamma(n+12)}{\Gamma(n)\sqrt{\lambda\varphi}} \quad (14)$$

This distance determines how far a node can transmit given a minimum required SNR at the receiver, i.e. the length of the longest possible hop for a given transmit power. For example, for PPP density $\lambda = 0.001$, the average distance to neighbours $n = \{1,10,20\}$ obtained from (14) with $m = 2$ and $\phi = \pi/4$ are $E[d_n] = \{32, 111, 158\}$ m, whereas for $\lambda = 0.01$ we obtain $E[d_n] = \{10, 35, 50\}$ m.

Fig. 1 plots the probability of connectivity versus the neighbour index n for PPP density $\lambda = \{1, 5, 10\} \cdot 10^{-3}$, assuming $\tilde{P}_t = 70$ dB, $m_s = 2$, path-loss exponents $\alpha = 3$ ($\ell = 3$, $k = 1$) and $\alpha = 3.5$ ($\ell = 7$, $k = 2$), and $\gamma_{th} = 5$ dB. The figure shows the impact of path-loss exponent α and PPP density λ on the probability of connectivity to the nth nearest neighbour. We note that assuming $N = -105$ dBm and $K = -35$ dB, $\tilde{P}_t = 70$ dB corresponds to $P_t = 70 + 35 - 105 = 0$ dBm.

In Fig. 2 we plot the probability of connectivity versus neighbour index n for an interference-limited system, that is, for a system where connectivity is based on the SIR association criterion. We assume the same system parameters with the previous plot but also consider $m_I = 1$ and INR $\Omega_I = 10$ dB for the interference signal. The plot shows the impact of interference on the probability that the nth nearest neighbour will successfully detect the transmission from the source node located at the origin.

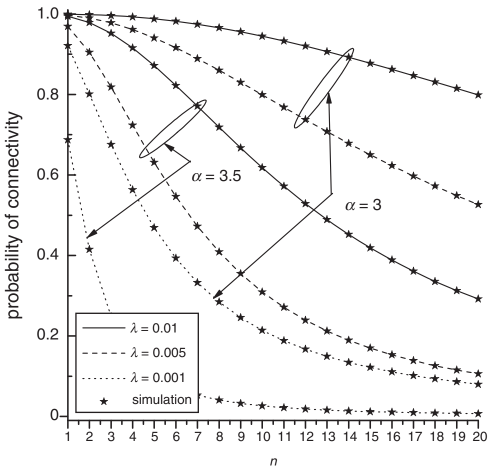

Fig. 2 *Probability of connectivity based on SIR versus neighbour index n*

Conclusion: In this Letter, we derived closed-form expressions for the PDF and CDF of the received SNR and received SIR at the nth nearest neighbour in a PPP random wireless network. The analytical expressions were used to investigate the impact of node density, transmit power, path-loss exponent, and association threshold on the probability that the nth nearest neighbour to a source node can successfully decode a transmitted packet.

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